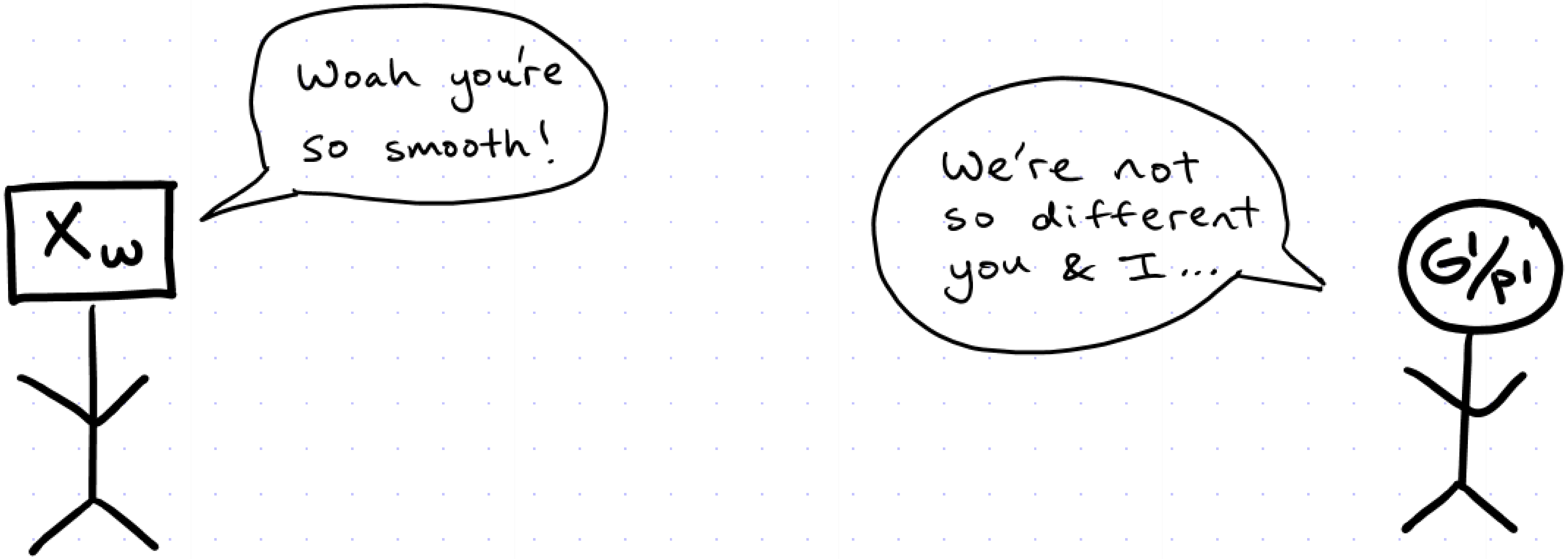
Smoothing Schubert classes by homogeneous subvarieties

Rachel Wu



Setting: homogeneous variety G/P

G complex semisimple linear algebraic group
$\cup$
P parabolic subgroup
$\cup$
B fixed Borel subgroup
$\cup$
H fixed maximal torus / Cartan

Weyl group $W := N(H)/H$, $W_P :=$ Weyl group associated to P

$W^P := \{$minimal length reps of cosets $W/W_P\}$

Schubert variety $X_w := \overline{BwP}/P \subset G/P$, $w \in W^P$

Schubert classes $[X_w]$ form a $\mathbb{Z}$-basis for $H_\bullet(G/P)$

Background & Motivation

[Borel-Haefliger 1961] "The rigidity problem"

In G/P, which X_w can be represented by a (non-Schubert) subvariety $Y \subset G/P$, ie. $[Y] = [X_w] \in H_\bullet(G/P)$?

Answered for:

- Grassmanians SL_n/P_i [Coskun 2011]
- Cominiscule G/P [Robles-The 2011] [Coskun-Robles 2013]
- Partial flag varieties (Type A, B, D) [Liu-Sheshmani-Yau 2024]

Questions [Coskun 2013]

Which $\mathbb{Z}_{\geq 0}$-linear combinations of Schubert classes $[X_w]$ in $H_\bullet(G/P)$ can be represented by a ...

- smooth subvariety?
- $\mathbb{Z}_{\geq 0}$-linear combination of classes of smooth subvarieties?
- linear combination of classes of smooth subvarieties?

Thm [Kollár–Voisin 2024] X smooth projective variety of dim n

If $d < \frac{n}{2}$, then for any cycle $z \in CH_d(X)$, there exist smooth subvarieties $Y_i \subset X$ and $a_i \in \mathbb{Z}$ such that $z = \sum a_i [Y_i]$.

Question

Which $\mathbb{Z}_{\geq 0}$-linear combinations of Schubert classes $[X_w] \in H_\bullet(G/P)$ can be represented by a $\mathbb{Z}_{\geq 0}$-linear combination of smooth subvarieties $Y_i \subset G/P$?

Currently working on SO_7/P_2 (Type B_3)

maximal parabolic associated to α_2

Strategy

① Find smooth subvarieties $Y \subset G/P$.

② Write $[Y] = \sum n_w [X_w] \in H_\bullet(G/P)$.

① Find smooth subvarieties $Y \subset G/P$.

- $\boxed{\text{homogeneous subvarieties}}$
- some Richardson varieties X_u^v
- some $H \cap X_w$, H general hyperplane in the minimal homogeneous embedding
- $H \cap H \cap \dots \cap H$

② Write $[Y] = \sum n_w [X_w]$ in $H_\bullet(G/P)$.

- $\boxed{\text{BGG polynomials \& divided difference operators}}$
- structure constants
- Chevalley / Pieri formula

① How do homogeneous subvarieties of G/P arise?

- subdiagrams of the Dynkin diagram
- sub root systems

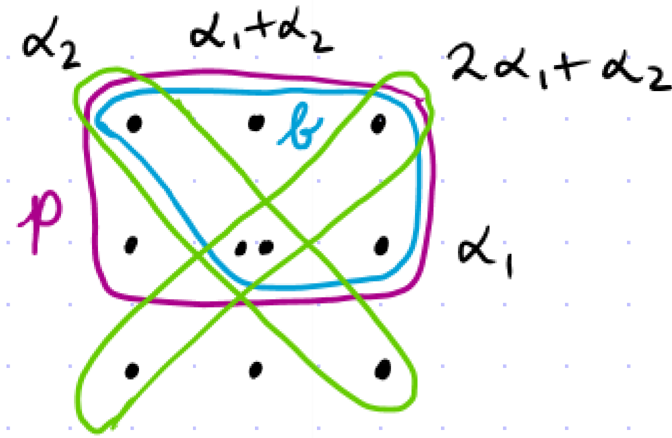
$\rightsquigarrow$ sub Lie algebra $\mathfrak{g}' \subset \mathfrak{g} \rightsquigarrow$ subgroup $G' \subset G$

$\rightsquigarrow$ subvariety $G'/P \cap G'$

- Hodge subdomains

Example

(Type C_2)



$\mathbb{P}^1 \times \mathbb{P}^1 \subset Sp_4/P_2$

② Given $G'/P' \subset G/P$, how do we write $[G'/P'] = \sum n_w [X_w] \in H_\bullet(G/P)$?

Start in G/B: Let $R := \mathbb{Q}[\alpha_1, \dots, \alpha_n]$

[Borel] $\phi : R/J \xrightarrow{\cong} H^\bullet(G/B)$

[Bernstein - Gelfand - Gelfand]

$\exists$ polynomials $\{P_w \mid w \in W\} \subset R/J$ dual to the

Schubert classes $\{[X_v] \mid v \in W\} \subset H_\bullet(G/B)$,

ie. $\delta_{wv} = \langle \underbrace{\phi(P_w)}_{\in H^\bullet(G/P)}, \underbrace{[X_v]}_{\in H_\bullet(G/P)} \rangle$

☆ compute P_w using divided difference operators: $A_i : R \to R$

$$f \mapsto \frac{f - s_i(f)}{\alpha_i}$$

[B-G-G] **Thm** $P_{w_0} = \frac{1}{|W|} \prod_{\alpha \in \Delta^+} \alpha$, $P_w = A_{w^{-1}w_0} P_{w_0}$

Thm $\langle \phi(f), [X_v] \rangle = (A_v f)(0)$ in G/B

Want

dim d

$$[G'/P'] = \sum_{\substack{w \in W^P \\ \ell(w) = d}} n_w [X_w] \in H_\bullet(G/P), \quad n_w = \langle Y_w , [G'/P'] \rangle$$

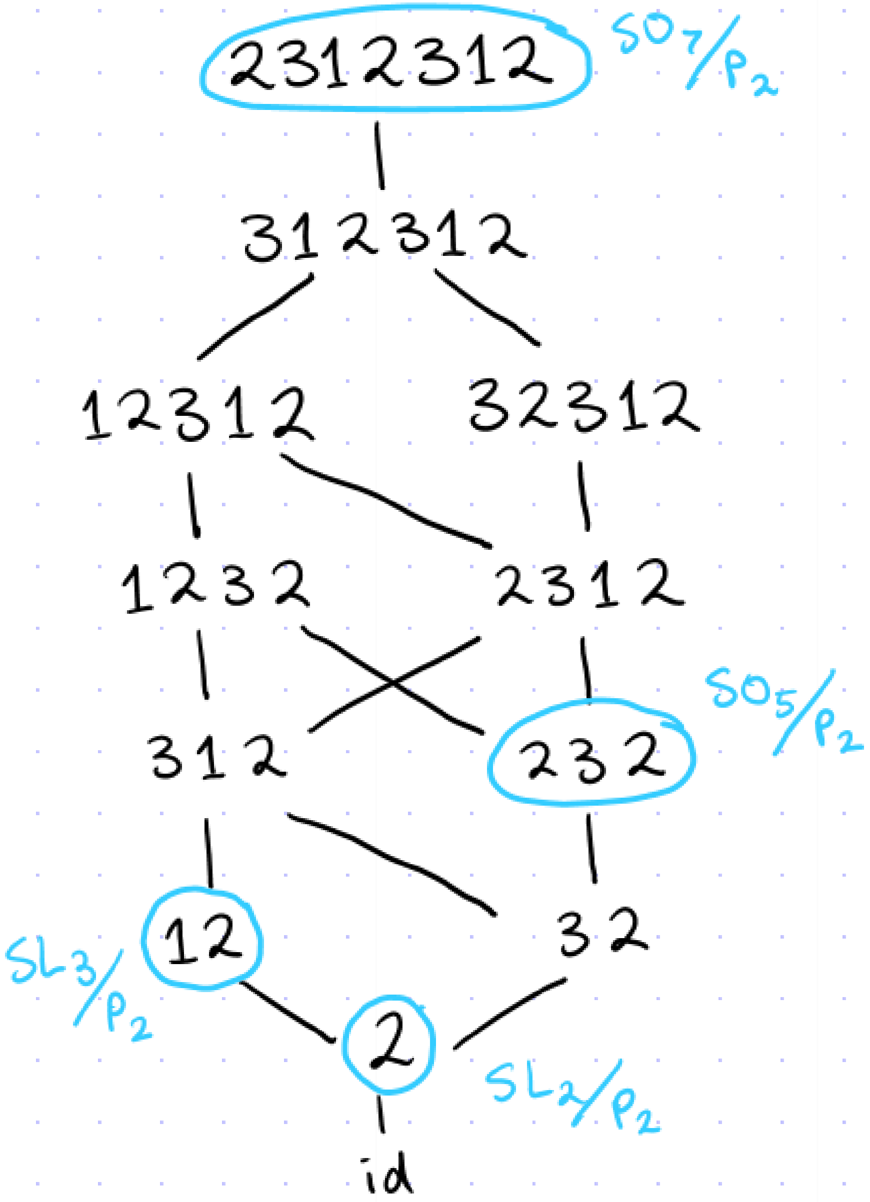
cohomology class dual to $[X_w]$

Proposition [Kerr-W.]

$$n_w = \left(A_{w_0'} \, P_w \big|_{\mathfrak{h}'}\right)(0)$$

longest element in $W^{P'}$

Hasse diagram



Computations in $H_\bullet(SO_7/P_2)$

(2,3,2)

$$[SL_2/P_2] = [X_2]$$

$$[SL_3/P_2] = [X_{12}], \quad [H \cap X_{232}] = [X_{32}]$$

(1,3,1) $$[SO_5/P_2] = [X_{232}]$$

(2,1,2) $$[SO_5/P_2] = 2[X_{312}]$$

$$[A_2/P_2] = [X_{312}] + [X_{232}]$$

$$[H^3] = 4[X_{2312}] + 2[X_{1232}]$$

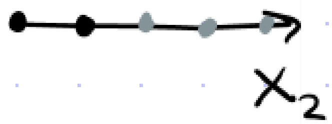
(2,2,2) $$[SO_6/P_2] = [X_{23212}]$$

$$[G_2/P_2] = [X_{12312}]$$

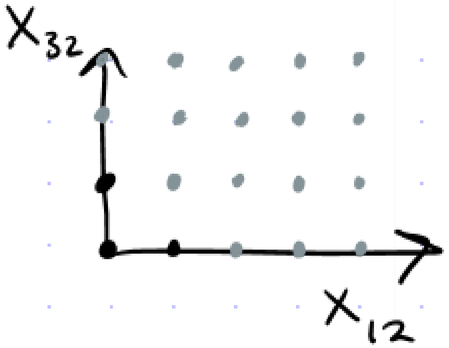
Which $\mathbb{Z}_{\geq 0}$-linear combinations of Schubert classes $[X_w] \in H_\bullet(G/P)$ can be represented by a $\mathbb{Z}_{\geq 0}$-linear combination of smooth subvarieties $Y_i \subset G/P$?

Results for $SO_7/P_2 = OG(2,7)$:

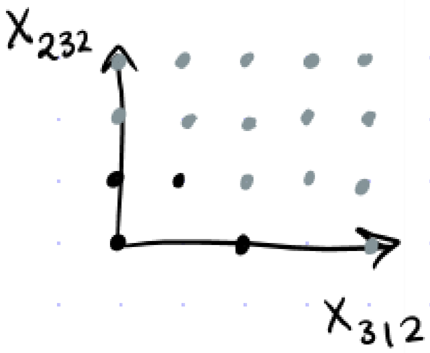
Dim 1



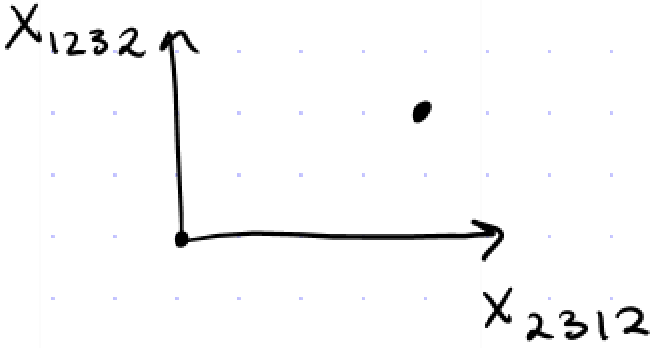
Dim 2



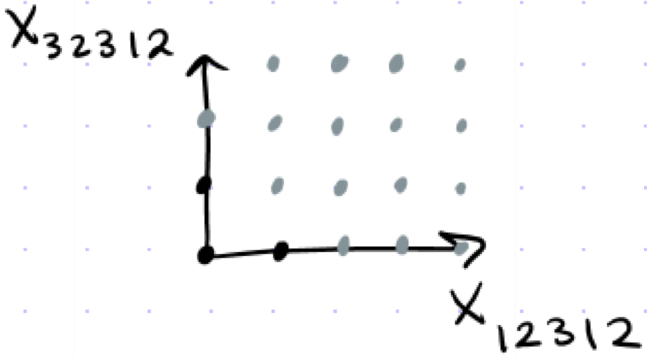
Dim 3



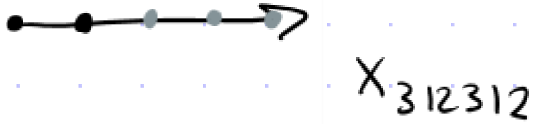
Dim 4



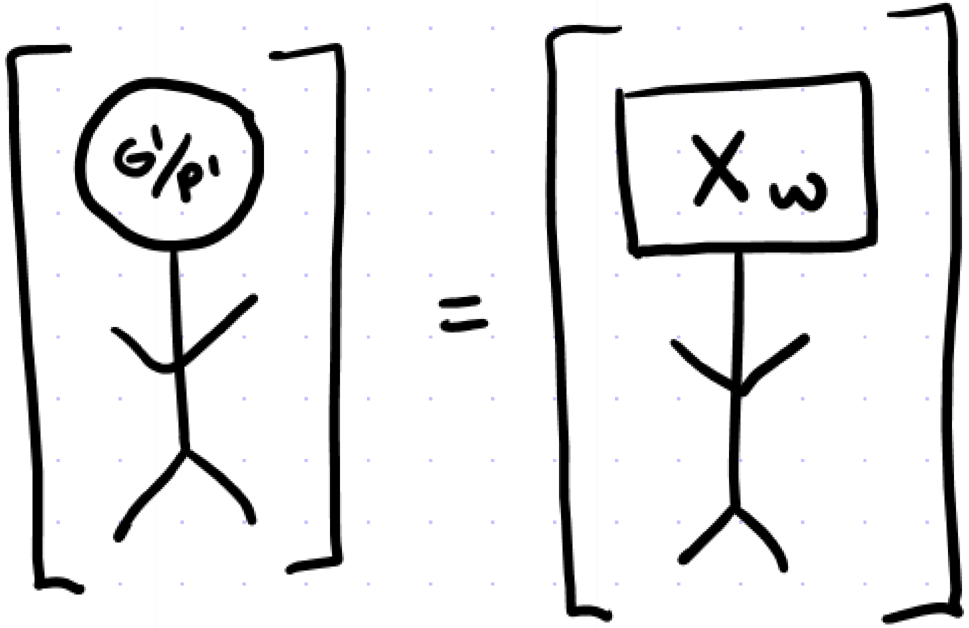
Dim 5



Dim 6



Thank you!

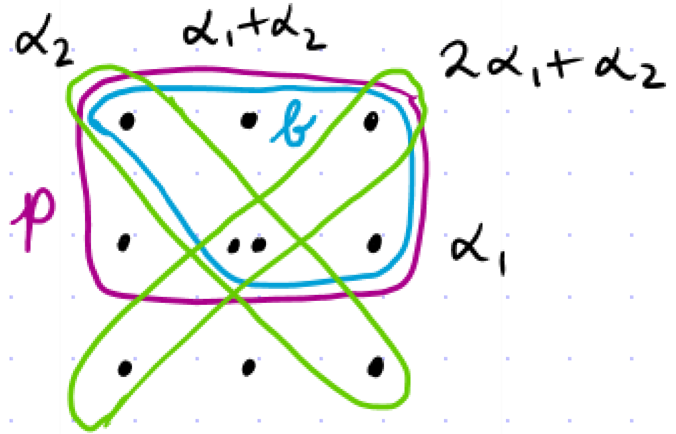


rachelwu@wustl.edu

Example

$\mathbb{P}^1 \times \mathbb{P}^1 \subset Sp_4/P$

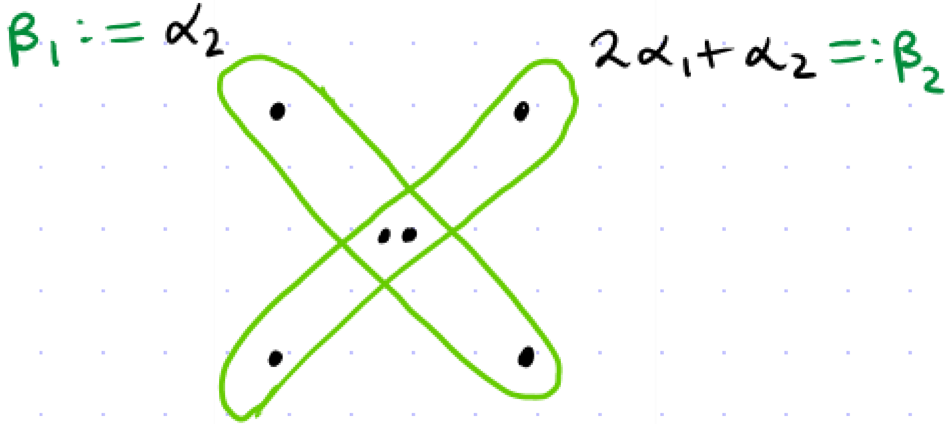
dim 2   dim 3



$$[\mathbb{P}^1 \times \mathbb{P}^1] = n_{12}[X_{12}]$$

$$P_{12} = \frac{1}{2}(\alpha_1+\alpha_2)^2$$

$$\Rightarrow P_{12}|_{\mathfrak{h}'} = \frac{1}{2}\left(\frac{\beta_2-\beta_1}{2} + \beta_1\right)^2$$



$$n_{12} = A_{\beta_1\beta_2} P_{12}|_{\mathfrak{h}'} = A_{\beta_1} A_{\beta_2} \frac{1}{8}(\beta_1+\beta_2)^2 = \ldots = 1$$

Now in G/P,

Let $W_P :=$ Weyl group associated to P.

Then $W^P := \{$minimal length reps of cosets $W/W_P\}$

indexes the Schubert classes in $H_\bullet(G/P)$.

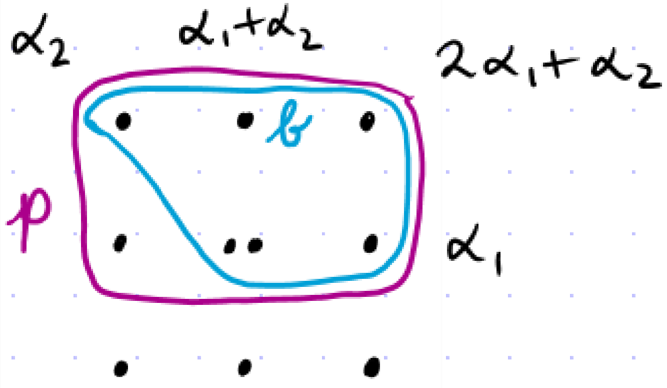
Also, $\pi_*([\tilde{X}_w]) = [X_w]$ if $w \in W^P$, $\pi_*([\tilde{X}_w]) = 0$ otherwise.

Example

$W = \{id, 1, 2, 12, 21, 121, 212, 1212\}$

$W_P = \langle 1 \rangle$

$W^P = W/W_P = \{id, 2, 12, 212\}$



Want

$$\overset{\dim d}{[G'/P']} = \sum_{\substack{w \in W^P \\ \ell(w) = d}} n_w [X_w], \quad n_w = \langle Y_w, [G'/P'] \rangle$$

cohomology class dual to $[X_w]$

Proposition [Kerr-W.]

$$n_w = \left(A_{w_0'} \, P_w|_{\mathfrak{h}'}\right)(0)$$

longest element in $W_{P'}$

$$\begin{tikzcd}
P_w \arrow[d, mapsto] & \in R/J \arrow[r, "\phi"] \arrow[d, "|_{\mathfrak{h}'}"] & H^\bullet(G/B) \arrow[d, "i^*"] \\
P_w|_{\mathfrak{h}'} & \in R'/J' \arrow[r, "\phi'"] & H^\bullet(G'/B')
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{X}_{w_0'} \subset G'/B' \arrow[r, hook, "i"] \arrow[d] & G/B \arrow[d, "\pi"] \\
X_{w_0'} = G'/P' \arrow[r, hook] & G/P
\end{tikzcd}$$

longest element in $W_{P'}$

③ Which Schubert varieties are smooth?

Thm [Hong-Mok]

P_i maximal parabolic associated to a long root α_i

$$\left\{ \begin{array}{l} \text{smooth} \\ \text{Schubert} \\ \text{varieties} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{connected subdiagrams} \\ \text{of the Dynkin diagram} \\ \text{containing } i\text{-th node} \end{array} \right\}$$

What is known?

Grassmanians SL_n/P_i

- [Coskun 2011] characterizes rigid Schubert classes & gives nearly sharp criterion for smoothability.

Cominiscule G/P

- [Robles-The 2011] identifies first order obstructions to multi-rigidity
- [Coskun-Robles 2013] shows obstructed classes are flexible

Partial flag varieties (Type A, B, D)

- [Liu-Sheshmani-Yau 2024] characterizes multi-rigidity